

## THE DIFFERENTIAL OPERATORS

LIST OF FORMULAS OF DIFFERENTIAL OPERATORS.

$$\textcircled{1} \vec{\nabla} = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \text{ (Nabla or del)}$$

\textcircled{2} Divergence of a vector. (Divergence dot)

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$\text{where } \vec{F} = [F_1, F_2, F_3]$$

$$\textcircled{3} \text{div}(\vec{F} \pm \vec{g}) = \text{div} \vec{F} \pm \text{div} \vec{g}$$

$$\text{div}(c\vec{F}) = c \text{div} \vec{F}$$

\textcircled{4} Laplace operator:  $\nabla^2$  is called Laplace operator.

$$\text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

\textcircled{5} Laplacian of a scalar function  $F$ .

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

\textcircled{6} Laplace's equation

$$\nabla^2 F = 0 \text{ is called Laplace equation.}$$

### ⑦ Harmonic function

If a function satisfies Laplace's equation it is called Harmonic function.

⑧ Gradient of a scalar function  $F = \vec{\nabla} F$  (scalar product)

$$= \frac{\partial F}{\partial x} \vec{i} + \frac{\partial F}{\partial y} \vec{j} + \frac{\partial F}{\partial z} \vec{k}$$

$$= \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]$$

### ⑨ Directional derivative

Derivative of a scalar function in the direction of a vector  $\vec{a}$ .

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\text{grad } F)$$

⑩ Level surface: The set of points satisfying

⑪ Unit normal vector to a level surface at the point.

$$\hat{n} = \frac{(\text{grad})_p}{|\text{grad}|_p}$$

$$\text{⑫ } \text{grad}(Fg) = F \text{ grad } g + g \text{ grad } F$$

$$\text{⑬ } \text{grad} \left( \frac{F}{g} \right) = \frac{g \text{ grad } F - F \text{ grad } g}{g^2}$$

$$\text{⑭ } \text{grad } r^n = n r^{n-2} \vec{r}$$

## The Differential operators CH # 5

This chapter deals with the differential operators like gradient, divergence and Curl.

The vectors differential operator  $\vec{\nabla}$   
(Read as "nabla" or "del").

$$\vec{\nabla} = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

$$\text{or } \vec{\nabla} = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

### Divergence of a vector

$$\text{let } \vec{F} = [F_1, F_2, F_3]$$

Then the scalar  $\vec{\nabla} \cdot \vec{F}$  is called divergence of  $\vec{F}$

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [F_1, F_2, F_3]$$

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

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Ex-1  
80 Calculate  $\text{div } \vec{F}$   $\varphi$

$$\vec{F}(x, y, z) = e^{xy} \cos z \vec{i} + e^{yz} \cos x \vec{j} + e^{xz} \cos y \vec{k}$$

Sol:-  $\vec{F}(x, y, z) = e^{xy} \cos z \vec{i} + e^{yz} \cos x \vec{j} + e^{xz} \cos y \vec{k}$

Here  $f_1 = e^{xy} \cos z$ ,  $f_2 = e^{yz} \cos x$ ,  $f_3 = e^{xz} \cos y$ .

$$\frac{\partial f_1}{\partial x} = e^{xy} \cos z \cdot y, \quad \frac{\partial f_2}{\partial y} = e^{yz} \cos x \cdot z, \quad \frac{\partial f_3}{\partial z} = e^{xz} \cos y \cdot x$$

$$\text{div } \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{div } \vec{F} = y e^{xy} \cos z + z e^{yz} \cos x + x e^{xz} \cos y. \text{ Ans}$$

### EXERCISE # 5.2

Q1  
80 Calculate the divergence of the following vector fields.

$$\vec{F} = (x^2 - xy^2) \vec{i} + (y^2 - yz^2) \vec{j} + (z^2 - zx^2) \vec{k}$$

Sol:- Here

$$\begin{array}{l} f_1 = x^2 - xy^2 \\ \frac{\partial f_1}{\partial x} = 2x - y^2 \end{array} \left| \begin{array}{l} f_2 = y^2 - yz^2 \\ \frac{\partial f_2}{\partial y} = 2y - z^2 \end{array} \right. \left. \begin{array}{l} f_3 = z^2 - zx^2 \\ \frac{\partial f_3}{\partial z} = 2z - x^2 \end{array} \right.$$

$$\text{div } \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{div } \vec{F} = 2x - y^2 + 2y - z^2 + 2z - x^2 = 2(x+y+z) - x^2 - y^2 - z^2$$

$$\text{div } \vec{F} = 2(x+y+z) - (x^2 + y^2 + z^2) \text{ Ans}$$

Q2/80 Calculate the divergence of the following vector

Field.  $\vec{F} = x^2 y z \vec{i} + x y^2 z \vec{j} + x y z^2 \vec{k}$ .

Sol:- Here

$$F_1 = x^2 y z \quad \left| \quad F_2 = x y^2 z \quad \left| \quad F_3 = x y z^2 \right. \right.$$

$$\frac{\partial F_1}{\partial x} = 2 x y z \quad \left| \quad \frac{\partial F_2}{\partial y} = 2 x y z \quad \left| \quad \frac{\partial F_3}{\partial z} = 2 x y z \right. \right.$$

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2 x y z + 2 x y z + 2 x y z$$

$\text{div } \vec{F} = 6 x y z$  Ans

Q3/80 Calculate the divergence of the following vector fields.  $\vec{F} = x \sin x y \vec{i} + y \sin y z \vec{j} + z \sin z x \vec{k}$ .

Sol:- Here

$$F_1 = x \sin x y \quad \left( \quad F_2 = y \sin y z \quad \left( \quad F_3 = z \sin z x \right. \right.$$

$$\frac{\partial F_1}{\partial x} = x(\cos x y) \cdot y + \sin x y \quad \left( \quad \frac{\partial F_2}{\partial y} = y(\cos y z) \cdot z + \sin y z \quad \left( \quad \frac{\partial F_3}{\partial z} = z(\cos z x) \cdot x + \sin z x \right. \right.$$

$$\frac{\partial F_1}{\partial x} = x y \cos x y + \sin x y \quad \left( \quad \frac{\partial F_2}{\partial y} = y z \cos y z + \sin y z \quad \left( \quad \frac{\partial F_3}{\partial z} = z x \cos z x + \sin z x \right. \right.$$

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= (x y \cos x y + \sin x y) + (y z \cos y z + \sin y z) + z x \cos z x + \sin z x$$

$$\text{div } \vec{F} = x y \cos x y + y z \cos y z + z x \cos z x + \sin x y + \sin y z + \sin z x$$

Ans

(Q4) Calculate the divergence of the following vector field

$$\vec{F} = \frac{x}{x+y+z} \vec{i} + \frac{y}{x+y+z} \vec{j} + \frac{z}{x+y+z} \vec{k}$$

Sol:- Here  $f_1 = \frac{x}{x+y+z}$

$$\frac{\partial f_1}{\partial x} = \frac{(x+y+z) \cdot 1 - x(1)}{(x+y+z)^2} = \frac{x+y+z-x}{(x+y+z)^2} = \frac{y+z}{(x+y+z)^2}$$

$$\boxed{\frac{\partial f_1}{\partial x} = \frac{y+z}{(x+y+z)^2}}$$

$$f_2 = \frac{y}{x+y+z}$$

$$\frac{\partial f_2}{\partial y} = \frac{(x+y+z) \cdot 1 - y(1)}{(x+y+z)^2} = \frac{x+y+z-y}{(x+y+z)^2} = \frac{x+z}{(x+y+z)^2}$$

$$\boxed{\frac{\partial f_2}{\partial y} = \frac{x+z}{(x+y+z)^2}}$$

$$f_3 = \frac{z}{x+y+z}$$

$$\frac{\partial f_3}{\partial z} = \frac{(x+y+z) \cdot 1 - z(1)}{(x+y+z)^2} = \frac{x+y+z-z}{(x+y+z)^2}$$

$$\boxed{\frac{\partial f_3}{\partial z} = \frac{x+y}{(x+y+z)^2}}$$

$$\text{div } \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{y+z}{(x+y+z)^2} + \frac{x+z}{(x+y+z)^2} + \frac{x+y}{(x+y+z)^2} = \frac{y+z+x+z+x+y}{(x+y+z)^2}$$

P.T.O

$$= \frac{2x+2y+2z}{(x+y+z)^2} = \frac{2(x+y+z)}{(x+y+z)^2} = \frac{2}{(x+y+z)}$$

$$\boxed{\operatorname{div} \vec{F} = \frac{2}{x+y+z}} \quad \text{Ans}$$

(Q5) Calculate the divergence of the vector field

$$\vec{F} = (x-y)^2 \vec{i} + (y-z)^2 \vec{j} + (z-x)^2 \vec{k}$$

$$\text{Sol: } f_1 = (x-y)^2 \quad f_2 = (y-z)^2 \quad f_3 = (z-x)^2$$

$$\frac{\partial f_1}{\partial x} = 2(x-y) \cdot 1 \quad \frac{\partial f_2}{\partial y} = 2(y-z) \cdot 1 \quad \frac{\partial f_3}{\partial z} = 2(z-x) \cdot 1$$

$$\boxed{\frac{\partial f_1}{\partial x} = 2(x-y)} \quad \boxed{\frac{\partial f_2}{\partial y} = 2(y-z)} \quad \boxed{\frac{\partial f_3}{\partial z} = 2(z-x)}$$

$$\operatorname{div} \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div} \vec{F} = 2(x-y) + 2(y-z) + 2(z-x)$$

$$\operatorname{div} \vec{F} = 2x - 2y + 2y - 2z + 2z - 2x$$

$$\boxed{\operatorname{div} \vec{F} = 0} \quad \text{Ans}$$

Q6/80 Ch-05

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Calculate the divergence of the vector field.

$$\vec{F} = \frac{z-x}{(x^2+y^2+z^2)} \vec{i} + \frac{x-y}{(x^2+y^2+z^2)} \vec{j} + \frac{y-z}{(x^2+y^2+z^2)} \vec{k}$$

Sol:-  $f_1 = \frac{z-x}{x^2+y^2+z^2}$

$$\frac{\partial f_1}{\partial x} = \frac{x^2+y^2+z^2(-1) - (z-x)(2x)}{(x^2+y^2+z^2)^2} = \frac{-x^2-y^2-z^2-2zx+2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f_1}{\partial x} = \frac{x^2-y^2-z^2-2zx}{(x^2+y^2+z^2)^2}$$

$$f_2 = \frac{x-y}{x^2+y^2+z^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{(x^2+y^2+z^2)(-1) - (x-y)2y}{(x^2+y^2+z^2)^2} = \frac{-x^2-y^2-z^2-2xy+2y^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{-x^2+y^2-z^2-2xy}{(x^2+y^2+z^2)^2}$$

$$f_3 = \frac{y-z}{x^2+y^2+z^2}$$

$$\frac{\partial f_3}{\partial z} = \frac{(x^2+y^2+z^2)(-1) - (y-z)(2z)}{(x^2+y^2+z^2)^2} = \frac{-x^2-y^2-z^2-2yz+2z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f_3}{\partial z} = \frac{-x^2-y^2+z^2-2yz}{(x^2+y^2+z^2)^2}$$

$$\text{div } \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{x^2-y^2-z^2-2zx}{(x^2+y^2+z^2)^2} + \frac{-x^2+y^2-2xy-z^2}{(x^2+y^2+z^2)^2} + \frac{-x^2-y^2+z^2-2yz}{(x^2+y^2+z^2)^2}$$



(9)

$$= \frac{x^2 - y^2 - z^2 - 2zx - x^2 + y^2 - z^2 - 2xy - x^2 - y^2 + z^2 - 2yz}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{-x^2 - y^2 - z^2 - 2zx - 2xy - 2yz}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{-(x^2 + y^2 + z^2 + 2zx + 2xy + 2yz)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{-(x+y+z)^2}{(x^2 + y^2 + z^2)^2} = - \left[ \frac{x+y+z}{x^2 + y^2 + z^2} \right]^2$$

$$\text{div } \vec{F} = - \left[ \frac{(x+y+z)}{x^2 + y^2 + z^2} \right]^2 \quad \text{Ans}$$

Available at  
www.mathcity.org

(Q7/80) Find divergence of the vector field.

$$\vec{F} = e^{xy} (\cos xy \vec{i} + \sin yz \vec{j} + z \sin zx \vec{k})$$

$$\text{Sol: } \vec{F}_1 = e^{xy} \cos xy \vec{i} + e^{xy} \sin yz \vec{j} + e^{xy} z \sin zx \vec{k}$$

$$\vec{F}_1 = e^{xy} \cos xy$$

$$\frac{\partial F_1}{\partial x} = e^{xy} (-\sin xy \cdot y) + (\cos xy) \cdot e^{xy} \cdot y$$

$$F_2 = e^{xy} \sin yz$$

$$\frac{\partial F_2}{\partial y} = e^{xy} (\cos yz \cdot z) + \sin yz \cdot e^{xy} \cdot x$$

$$F_3 = e^{xy} z \sin zx \cdot (\text{Note this point})$$

$$\frac{\partial F_3}{\partial z} = e^{xy} [zx \cos zx + \sin zx \cdot 1]$$

$$\operatorname{div} \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div} \vec{F} = -y e^{xy} \sin xy + y \cos y e^{xy} + z e^{xy} \cos yz \\ + x \sin yz e^{xy} + z x e^{xy} \cos zx + e^{xy} \sin zx$$

$$\operatorname{div} \vec{F} = e^{xy} (-y \sin xy + y \cos xy + z \cos yz + z x \cos zx + \sin zx) \quad \text{Ans}$$

(i) Prove  $\operatorname{div}(\vec{F} + \vec{g}) = \operatorname{div} \vec{F} + \operatorname{div} \vec{g}$ .

Sol:- let  $\vec{F} = [F_1, F_2, F_3]$ ,  $\vec{g} = [g_1, g_2, g_3]$ .

$$\vec{F} + \vec{g} = [F_1 + g_1, F_2 + g_2, F_3 + g_3]$$

$$\operatorname{div}(\vec{F} + \vec{g}) = \operatorname{div}[F_1 + g_1, F_2 + g_2, F_3 + g_3]$$

$$= \nabla \cdot [F_1 + g_1, F_2 + g_2, F_3 + g_3]$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [F_1 + g_1, F_2 + g_2, F_3 + g_3]$$

$$= \frac{\partial}{\partial x} (F_1 + g_1) + \frac{\partial}{\partial y} (F_2 + g_2) + \frac{\partial}{\partial z} (F_3 + g_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial g_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial g_2}{\partial y} + \frac{\partial F_3}{\partial z} + \frac{\partial g_3}{\partial z}$$

$$= \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) + \left( \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z} \right)$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [F_1, F_2, F_3] + \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [g_1, g_2, g_3]$$

$$= \nabla \cdot \vec{F} + \nabla \cdot \vec{g}$$

$$\boxed{\text{div}(\vec{F} + \vec{g}) = \text{div} \vec{F} + \text{div} \vec{g}} \quad \text{Ans}$$

(Q8) Find  $\text{div}(\vec{F} + \vec{g})$  ?

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{and} \quad \vec{g} = x \sin x \vec{i} + y \cos y \vec{j} + z \sec z \vec{k}$$

Sol:-  $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

$$F_1 = x \quad F_2 = y \quad F_3 = z$$

$$\frac{\partial F_1}{\partial x} = 1 \quad \frac{\partial F_2}{\partial y} = 1 \quad \frac{\partial F_3}{\partial z} = 1$$

$$\text{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{div} \vec{F} = 1 + 1 + 1$$

$$\boxed{\text{div} \vec{F} = 3}$$

$$\vec{g} = x \sin x \vec{i} + y \cos y \vec{j} + z \sec z \vec{k}$$

$$g_1 = x \sin x$$

$$\frac{\partial g_1}{\partial x} = x \cos x + \sin x - 1$$

$$g_2 = y \cos y$$

$$\frac{\partial g_2}{\partial y} = y(-\sin y) + \cos y - 1$$

$$g_3 = z \sec z$$

$$\frac{\partial g_3}{\partial z} = z \cdot \sec z \tan z + \sec z - 1$$

$$\text{div} \vec{g} = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}$$

$$\text{div} \vec{g} = x \cos x + \sin x - 1 + (-y \sin y + \cos y - 1) + (z \sec z \tan z + \sec z - 1)$$

$$\text{div}(\vec{F} + \vec{g}) = \text{div} \vec{F} + \text{div} \vec{g}$$

$$\text{div}(\vec{F} + \vec{g}) = 3 + x \cos x + \sin x - y \sin y + \cos y + z \sec z \tan z + \sec z - 1$$

Ans.

## LAPLACE OPERATOR

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$\nabla^2 = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\nabla^2$  is called the Laplace operator.

## LAPLACIAN OF A SCALAR FUNCTION $F$

$\nabla^2 F$  is called Laplacian of  $F$

$$\text{Hence } \nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

## LAPLACE'S EQUATION

The equation  $\nabla^2 F = 0$  is known as Laplace's equation.

## HARMONIC FUNCTION

$$\text{If } \nabla^2 F = 0$$

Then the function  $F$  is said to be

harmonic.

Q10/81 Find the Laplacian of F if

$$(a) F(x, y, z) = x^2yz + x(y^2z + xy z^2)$$

$$\text{Sol: } F(x, y, z) = x^2yz + x(y^2z + xy z^2)$$

$$\frac{\partial F}{\partial x} = 2xyz + y^2z + yz^2$$

$$\frac{\partial^2 F}{\partial x^2} = 2yz + 0 + 0 = 2yz$$

$$\boxed{\frac{\partial^2 F}{\partial x^2} = 2yz}$$

$$\frac{\partial F}{\partial y} = x^2z + 2xyz + xz^2$$

$$\frac{\partial^2 F}{\partial y^2} = 0 + 2xz + 0 = 2xz$$

$$\boxed{\frac{\partial^2 F}{\partial y^2} = 2xz}$$

$$\frac{\partial F}{\partial z} = x^2y + xy^2 + 2xy z$$

$$\frac{\partial^2 F}{\partial z^2} = 0 + 0 + 2xy$$

$$\boxed{\frac{\partial^2 F}{\partial z^2} = 2xy}$$

Laplacian of F

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

$$\nabla^2 F = 2yz + 2xz + 2xy$$

$$\boxed{\nabla^2 F = 2(yz + xz + xy)} \quad \text{Ans}$$

$$(b) F(x, y, z) = yz \cos x + zx \cos y + xy \cos z$$

$$\text{Sol: } \nabla^2 F = ?$$

Since we know that

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} \quad \text{--- (1)}$$

$$\text{So, } \frac{\partial F}{\partial x} = yz(-\sin x) + z(1) \cos y + x(1) \cos z$$

$$\frac{\partial^2 F}{\partial x^2} = -yz \cos x + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 F}{\partial x^2} = -yz \cos x}$$

$$\frac{\partial F}{\partial y} = (1) z \cos x + zx(-\sin y) + x(1) \cos z$$

$$\frac{\partial^2 F}{\partial y^2} = 0 - zx \cos y + 0 \Rightarrow \boxed{\frac{\partial^2 F}{\partial y^2} = -zx \cos y}$$

$$\frac{\partial F}{\partial z} = y(1) \cos x + (1) x \cos y + xy(-\sin z)$$

$$\frac{\partial^2 F}{\partial z^2} = 0 + 0 - xy \cos z \Rightarrow \boxed{\frac{\partial^2 F}{\partial z^2} = -xy \cos z}$$

Putting all these three values in equation (1)

$$\nabla^2 F = -yz \cos x - zx \cos y - xy \cos z$$

$$\boxed{\nabla^2 F = -(yz \cos x + zx \cos y + xy \cos z)}$$

Ans

(14)

Ch-05

(Q11/81) If  $f = a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2zx + b_3xy$ .  
 where  $a_1, a_2, a_3$  and  $b_1, b_2, b_3$  are all constants.  
 Show that the Laplacian of  $F$  is zero. (may be asked as constant)

Sol:-  $f = a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2zx + b_3xy$

$$\frac{\partial f}{\partial x} = 2a_1x + 0 + 0 + 0 + b_2z + b_3y$$

$$\frac{\partial^2 f}{\partial x^2} = 2a_1 + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = 2a_1}$$

$$\frac{\partial f}{\partial y} = 0 + 2a_2y + 0 + 0 + 0 + b_3x$$

$$\frac{\partial^2 f}{\partial y^2} = 2a_2 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial y^2} = 2a_2}$$

$$\frac{\partial f}{\partial z} = 0 + 0 + 2a_3z + 0 + 0$$

$$\frac{\partial^2 f}{\partial z^2} = 2a_3 + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial z^2} = 2a_3}$$

Laplacian of  $f$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 2a_1 + 2a_2 + 2a_3$$

$$\boxed{\nabla^2 f = 2(a_1 + a_2 + a_3)} \quad \text{Ans}$$

## GRADIENT OF A SCALAR FUNCTION

$\vec{\nabla}F$  is called gradient of the scalar function  $F$ .

$$\text{grad } F = \vec{\nabla}F \quad \text{where} \quad \vec{\nabla} = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

v.v. imp  
Q.12  
81

Prove that  $\text{div}(F\vec{\nabla}g) = F\nabla^2g + \vec{\nabla}F \cdot \vec{\nabla}g$

Sol:- L.H.S

$$\text{div}(F\vec{\nabla}g) = \text{div} \left[ F \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] g \right]$$

$$= \text{div} \left[ F \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right] \right]$$

$$= \text{div} \left[ F \frac{\partial g}{\partial x}, F \frac{\partial g}{\partial y}, F \frac{\partial g}{\partial z} \right]$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \left[ F \frac{\partial g}{\partial x}, F \frac{\partial g}{\partial y}, F \frac{\partial g}{\partial z} \right]$$

$$= \frac{\partial}{\partial x} \left( F \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y} \left( F \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial z} \left( F \frac{\partial g}{\partial z} \right)$$

$$\text{div}(F\vec{\nabla}g) = F \frac{\partial^2 g}{\partial x^2} + \frac{\partial F}{\partial x} \frac{\partial g}{\partial x} + F \frac{\partial^2 g}{\partial y^2} + \frac{\partial F}{\partial y} \frac{\partial g}{\partial y} + F \frac{\partial^2 g}{\partial z^2} + \frac{\partial F}{\partial z} \frac{\partial g}{\partial z}$$

R.H.S =

$$F(\nabla^2g) + \vec{\nabla}F \cdot \vec{\nabla}g = F \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] \cdot \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right]$$

$$F(\nabla^2g) + \vec{\nabla}F \cdot \vec{\nabla}g = F \frac{\partial^2 g}{\partial x^2} + F \frac{\partial^2 g}{\partial y^2} + F \frac{\partial^2 g}{\partial z^2} + \frac{\partial F}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial g}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial g}{\partial z}$$

① = ②

∴  $\text{div}(F\vec{\nabla}g) = F\nabla^2g + \vec{\nabla}F \cdot \vec{\nabla}g$  proved



Q13  
81

If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $\vec{a}$  is a constant prove

$$(i) \operatorname{div}(\vec{a} \times \vec{r}) = 0$$

$$\text{Sol: } \operatorname{div}(\vec{a} \times \vec{r}) = \vec{\nabla} \cdot (\vec{a} \times \vec{r})$$

$$= \vec{\nabla} \cdot \vec{a} \times \vec{r} \quad (\text{scalar triple product})$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}, \text{ where } \vec{a} = [a_1, a_2, a_3]$$

Expanding by R,

$$= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 - 0 + 0 - 0 + 0 - 0$$

$$\boxed{\operatorname{div}(\vec{a} \times \vec{r}) = 0} \quad \text{proved}$$

$$\text{V-imp (ii) } \operatorname{div}[\vec{a} \times (\vec{r} \times \vec{a})] = 2a^2$$

$$\text{Sol: } \operatorname{div}[\vec{a} \times (\vec{r} \times \vec{a})] = \operatorname{div}[(\vec{a} \cdot \vec{a})\vec{r} - (\vec{a} \cdot \vec{r})\vec{a}]$$

(vector triple product)

$$= \operatorname{div}[a^2 \vec{r} - (\vec{a} \cdot \vec{r})\vec{a}]$$

$$= \operatorname{div} a^2 \vec{r} - \operatorname{div}(\vec{a} \cdot \vec{r})\vec{a}$$

$$= a^2 \operatorname{div} \vec{r} - \operatorname{div}(\vec{a} \cdot \vec{r})\vec{a} \quad \text{--- (1)}$$

$$\operatorname{div} \vec{r} = \vec{\nabla} \cdot \vec{r}$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [x, y, z]$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1$$

$$\operatorname{div} \vec{r} = \vec{\nabla} \cdot \vec{r} = 3.$$

$$(\vec{a} \cdot \vec{r}) \vec{a} = ([a_1, a_2, a_3] \cdot [x, y, z]) [a_1, a_2, a_3]$$

$$= (a_1 x + a_2 y + a_3 z) [a_1, a_2, a_3]$$

$$= [(a_1 x + a_2 y + a_3 z) a_1, (a_1 x + a_2 y + a_3 z) a_2, (a_1 x + a_2 y + a_3 z) a_3]$$

$$\operatorname{div} (\vec{a} \cdot \vec{r}) \vec{a} = \vec{\nabla} \cdot (\vec{a} \cdot \vec{r}) \vec{a}$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [(a_1 x + a_2 y + a_3 z) a_1, (a_1 x + a_2 y + a_3 z) a_2, (a_1 x + a_2 y + a_3 z) a_3]$$

$$= \frac{\partial}{\partial x} (a_1^2 x + a_1 a_2 y + a_1 a_3 z) + \frac{\partial}{\partial y} (a_1 x + a_2^2 y + a_1 a_3 z) + \frac{\partial}{\partial z} (a_3 a_1 x + a_2 a_3 y + a_3^2 z)$$

$$= a_1^2 + 0 + 0 + 0 + a_2^2 + 0 + 0 + 0 + a_3^2$$

$$= a_1^2 + a_2^2 + a_3^2 = \left( \sqrt{a_1^2 + a_2^2 + a_3^2} \right)^2$$

$$= |\vec{a}|^2$$

$$\operatorname{div} (\vec{a} \cdot \vec{r}) \vec{a} = a^2$$

$$\textcircled{1} \Rightarrow \operatorname{div} [\vec{a} \times (\vec{r} \times \vec{a})] = a^2(3) - a^2 = 3a^2 - a^2$$

$$\operatorname{div} [\vec{a} \times (\vec{r} \times \vec{a})] = 2a^2 \quad \underline{\text{Ans}}$$

Q15  
81 Show  $\vec{\nabla} \cdot (\vec{\nabla} F \times \vec{\nabla} g) = 0$

Sol:-  $\vec{\nabla} F = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] F$   
 $= \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]$

$\vec{\nabla} g = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] g$   
 $= \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right]$

$\vec{\nabla} \cdot (\vec{\nabla} F \times \vec{\nabla} g) = \vec{\nabla} \cdot \vec{\nabla} F \times \vec{\nabla} g$  (scalar triple product)

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}$$

Expanding  
by R<sub>1</sub>

$$= \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial F}{\partial z} \frac{\partial g}{\partial y} \right)$$

$$+ \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial g}{\partial z} \right)$$

$$+ \frac{\partial}{\partial z} \left( \frac{\partial F}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial g}{\partial x} \right)$$

$$= \left( \frac{\partial F}{\partial y} \right) \frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 F}{\partial x \partial y} \frac{\partial g}{\partial z} - \left( \frac{\partial F}{\partial z} \right) \frac{\partial^2 g}{\partial x \partial y}$$

$$- \frac{\partial^2 F}{\partial x \partial z} \frac{\partial g}{\partial y} + \left( \frac{\partial F}{\partial z} \right) \frac{\partial^2 g}{\partial y \partial x} + \frac{\partial^2 F}{\partial y \partial z} \frac{\partial g}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial^2 g}{\partial y \partial z}$$

$$- \frac{\partial^2 F \partial g}{\partial y \partial x \partial z} + \frac{\partial F}{\partial x} \frac{\partial^2 g}{\partial z \partial y} + \frac{\partial^2 F}{\partial z \partial x} \frac{\partial g}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial^2 g}{\partial z \partial x}$$

$$- \frac{\partial^2 F}{\partial z \partial y} \frac{\partial g}{\partial x}$$

$$\vec{\nabla} \cdot (\vec{\nabla} f \times \vec{\nabla} g) = 0 \quad \text{proved}$$

### DIRECTIONAL DERIVATIVES

Derivative of a scalar function in the direction of a vector  $\vec{a}$

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\text{grad } F)$$

#### EXERCISE #5.4

Find the derivative of  $F$  at  $p$  in direction of  $\vec{a}$ , where

Q1  
77  $F = e^{x+y+z}$ ,  $p(0,0,0)$ ,  $\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k}$

Sol:- Directional derivative of  $F$

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } F \quad \text{--- (1)}$$

$$F = e^{x+y+z}$$

$$\frac{\partial F}{\partial x} = e^{x+y+z} \Big|_{(0,0,0)} = e^{0+0+0} = e^0 = 1$$

$$\frac{\partial F}{\partial y} = e^{x+y+z} \Big|_{(0,0,0)} = e^{0+0+0} = e^0 = 1$$

$$\frac{\partial F}{\partial z} = e^{x+y+z} \Big|_{(0,0,0)} = e^{0+0+0} = 1$$

$$\text{grad } F = \vec{\nabla} F = \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] = [1, 1, 1]$$

$$\textcircled{1} \Rightarrow \frac{dF}{ds} = \frac{\vec{i} + 2\vec{j} - 2\vec{k}}{|\vec{i} + 2\vec{j} - 2\vec{k}|} \cdot [1, 1, 1]$$

$$= \frac{[1, 2, -2]}{\sqrt{1+4+4}} \cdot [1, 1, 1]$$

$$= \frac{1(1) + 2(1) + (-2)(1)}{3}$$

$$\boxed{\frac{dF}{ds} = \frac{1}{3}}$$

Ans

$$\textcircled{\frac{Q2}{77}} \quad F = e^{yz} \cos x + e^{zx} \cos y + e^{xy} \cos z$$

$$P\left(\frac{\pi}{6}, \frac{\pi}{3}, 0\right), \quad \vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$$

Sol:- Directional derivative of F

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } F \quad \text{--- ①}$$

$$F = e^{yz} \cos x + e^{zx} \cos y + e^{xy} \cos z$$

$$\frac{\partial F}{\partial x} = e^{yz} (-\sin x) + \cos y e^{zx} (z) + \cos z e^{xy} \cdot y$$

$$= e^{\frac{\pi}{3} \cdot 0} (-\sin \frac{\pi}{6}) + \cos \frac{\pi}{3} (e^{0 \cdot \frac{\pi}{6}} \cdot 0) + \cos 0 e^{\frac{\pi}{6} \cdot \frac{\pi}{3}}$$

$$= 1(-\frac{1}{2}) + (\frac{1}{2})(0) + 1 \cdot e^{\frac{\pi^2}{18}} \cdot \frac{\pi}{3}$$

$$\frac{\partial F}{\partial x} = -\frac{1}{2} + \frac{\pi}{3} e^{\frac{\pi^2}{18}}$$

$$\begin{aligned} \frac{\partial F}{\partial y} &= (e^{y^2} \cdot z) \cos x + e^{2x} (-\sin y) + (e^{xy} \cdot x) \cos z \\ &= (e^{\frac{\pi}{3} \cdot 0} \cdot 0) \cos \frac{\pi}{6} + e^{0 \cdot \frac{\pi}{6}} (-\sin \frac{\pi}{3}) + (e^{\frac{\pi}{6} \cdot \frac{\pi}{3}} \cdot \frac{\pi}{3}) \cos 0 \\ &= 1 \left( -\frac{\sqrt{3}}{2} \right) + \frac{\pi}{6} e^{\frac{\pi^2}{18}} \end{aligned}$$

$$\frac{\partial F}{\partial y} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6} e^{\frac{\pi^2}{18}}$$

$$\begin{aligned} \frac{\partial F}{\partial z} &= (e^{y^2} \cdot y) \cos x + (e^{2x} \cdot x) \cos y + e^{xy} (-\sin z) \\ &= (e^{\frac{\pi}{3} \cdot 0} \cdot \frac{\pi}{3}) \cos \frac{\pi}{6} + (e^{0 \cdot \frac{\pi}{6}} \cdot \frac{\pi}{6}) \cos \frac{\pi}{3} + e^{\frac{\pi}{6} \cdot \frac{\pi}{3}} (-\sin 0) \\ &= \frac{\pi}{3} \left( \frac{\sqrt{3}}{2} \right) + \frac{\pi}{6} \left( \frac{1}{2} \right) \end{aligned}$$

$$\frac{\partial F}{\partial z} = \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}$$

$$\begin{aligned} \text{grad } F &= \vec{\nabla} F = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] F \\ &= \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] \end{aligned}$$

$$= \left[ -\frac{1}{2} + \frac{\pi}{3} e^{\frac{\pi^2}{18}}, -\frac{\sqrt{3}}{2} + \frac{\pi}{6} e^{\frac{\pi^2}{18}}, \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12} \right]$$

$$= \frac{[3, 2, -1]}{\sqrt{3^2 + 2^2 + 1}} \cdot \left[ -\frac{1}{2} + \frac{\pi}{3} e^{\frac{\pi^2}{18}}, -\frac{\sqrt{3}}{2}, \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12} \right]$$

$$= \frac{1}{\sqrt{14}} \left[ -\frac{3}{2} + \pi e^{\frac{\pi^2}{18}} - \sqrt{3} + \frac{\pi}{3} e^{\frac{\pi^2}{18}} - \frac{\pi}{2\sqrt{3}} - \frac{\pi}{12} \right]$$

$$\frac{dF}{ds} = \frac{1}{\sqrt{14}} \left[ -\frac{3}{2} - \sqrt{3} + \pi e^{\frac{\pi^2}{18}} + \frac{\pi}{3} e^{\frac{\pi^2}{18}} - \pi \left( \frac{1}{2\sqrt{3}} + \frac{1}{12} \right) \right]$$

Ans

(Q3/78)  $F = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}$ ,  $P(1, -1, 1)$

$$\vec{a} = \vec{i} - \vec{j} + \vec{k}$$

Sol: Directional derivative is

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\text{grad } F) \quad \text{--- (1)}$$

$$F = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}$$

$$\frac{\partial F}{\partial x} = -\frac{yz}{x^2} + \frac{z}{y} + \frac{y}{z} = -\frac{(-1)(1)}{1^2} + \frac{1}{-1} + \frac{-1}{1}$$

$$\boxed{\frac{\partial F}{\partial x} = -1}$$

$$\frac{\partial F}{\partial y} = \frac{z}{x} - \frac{zx}{y^2} + \frac{x}{z} = \frac{1}{1} - \frac{1 \cdot 1}{(-1)^2} + \frac{1}{1}$$

$$\boxed{\frac{\partial F}{\partial y} = 1}$$

$$\frac{\partial F}{\partial z} = \frac{y}{x} + \frac{x}{y} - \frac{xy}{z^2} = \frac{-1}{1} + \frac{1}{-1} - \frac{1(-1)}{1^2}$$

$$\boxed{\frac{\partial F}{\partial z} = -1}$$

$$\text{grad } F = \vec{\nabla} F = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] F = \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]$$

$$\text{grad } F = [-1, 1, -1]$$

$$P \cdot T = 0$$

① ⇒

$$\frac{dF}{ds} = \frac{[1, -1, 1] \cdot [-1, 1, -1]}{\sqrt{1^2 + 1^2 + 1^2}}$$

$$= \frac{1(-1) + (-1)(1) + 1(-1)}{\sqrt{3}} = \frac{-3}{\sqrt{3}} = \frac{-\sqrt{3}\sqrt{3}}{\sqrt{3}}$$

$$\boxed{\frac{dF}{ds} = -\sqrt{3}} \quad \text{Ans}$$

Q4)  $f = x^2 + y^2 + z^2$ ,  $P(2, 2, 2)$ ,  $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$

Sol:  $\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } F$  — ①

$$F = x^2 + y^2 + z^2$$

$$\frac{\partial F}{\partial x} = 2x + 0 + 0 = 2(2) = 4$$

$$\frac{\partial F}{\partial y} = 0 + 2y + 0 = 2(2) = 4$$

$$\frac{\partial F}{\partial z} = 0 + 0 + 2z = 2(2) = 4$$

$$\text{grad } F = \vec{\nabla} F = \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] = [4, 4, 4]$$

$$\text{grad } F = [4, 4, 4]$$

$$S \quad \frac{dF}{ds} = \frac{\vec{i} + 2\vec{j} - 3\vec{k}}{\sqrt{1^2 + 2^2 + (-3)^2}} \cdot \text{grad } F$$

$$= \frac{(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot [4, 4, 4]}{\sqrt{1+4+9}} = \frac{4\vec{i} + 8\vec{j} - 12\vec{k}}{\sqrt{14}}$$



Q6/78  $F = x^2y^2 + 4yz^2$ ,  $P(1, -2, 1)$ ,  $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

Sol:-

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } F \quad \text{--- (1)}$$

Do?



Q11/78 Find the rate of change of  $F = x^2 + yz$  at  $(3, 1, -5)$  in the direction of  $\vec{i} + 2\vec{j} - \vec{k}$ .

Sol:- Here  $F = x^2 + yz$ ,  $P(3, 1, -5)$ ,  $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ .

Rate of change of  $F$  in the direction of vector  $\vec{a}$  is directional derivative.

$$\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } F \quad \text{--- (1)}$$

$$F = x^2 + yz$$

$$\frac{\partial F}{\partial x} = 2x = 2(3) = 6$$

$$\frac{\partial F}{\partial y} = z = -5$$

P.T.O

$$\frac{\partial F}{\partial z} = \gamma = 1$$

$$\begin{aligned} \text{grad } F &= \vec{\nabla} F = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] F \\ &= \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] = [6, -5, 1] \end{aligned}$$

①  $\Rightarrow$

$$\frac{dF}{ds} = \frac{[1, 2, -1] \cdot [6, -5, 1]}{\sqrt{1^2 + 2^2 + (-1)^2}}$$

$$= \frac{(1)(6) + (2)(-5) + (-1)(1)}{\sqrt{1+4+1}} = \frac{6-10-1}{\sqrt{6}}$$

$$\boxed{\frac{dF}{ds} = \frac{-5}{\sqrt{6}}} \quad \text{Ans}$$

Ex-1  
76 Find the directional derivative of the  $F = x^2 + yz$  in the direction of  $\vec{i} + 2\vec{j} - \vec{k}$  at the point  $P(3, 1, -5)$ .

Sol: Since  $\frac{dF}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } F$  ①

$$\text{grad } F = \vec{\nabla} F = \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]$$

$$\frac{\partial F}{\partial x} = 2x + 0 = 2(3) = 6$$

$$\frac{\partial F}{\partial y} = z = -5$$

$$\frac{\partial F}{\partial z} = 0 + y = 1$$

$$P = T \cdot 0$$

$$\begin{aligned} \frac{df}{ds} &= \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f \\ &= \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{1^2 + 2^2 + (-1)^2}} \cdot (6, -5, 1) \\ &= \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{1+4+1}} \cdot (6, -5, 1) \\ &= \frac{6\vec{i} - 10\vec{j} - 1\vec{k}}{\sqrt{6}} \quad \text{Ans} \end{aligned}$$

### Level Surface Of a Scalar Function.

Let us suppose that  $f(x, y, z)$  denote a scalar function; Then for a constant  $C$ , the set of points satisfying the equation  $f(x, y, z) = C$  is called a level surface of  $f$ .

Unit normal vector to a level surface at the point  $P$ .

$$\vec{n} = \frac{(\text{grad } f)_P}{|\text{grad } f|_P}$$

where  $(\text{grad } f)_P$  is called the given point.

Find the unit normal vector to the given level surface at the given point P.

(Q7)  $x^3 - y^3 + z^3 - 3xyz = 14$ ,  $P(2, 1, -1)$

Sol: Here  $f = x^3 - y^3 + z^3 - 3xyz$

$$\frac{\partial f}{\partial x} = 3x^2 - 0 + 0 - 3yz = 3(2)^2 - 3(1)(-1) = 15$$

$$\frac{\partial f}{\partial y} = 0 - 3y^2 + 0 - 3xz = -3(1)^2 - 3(2)(-1) = 3$$

$$\frac{\partial f}{\partial z} = 0 - 0 + 3z^2 - 3xy = 3(-1)^2 - 3 \cdot 2 \cdot 1 = -3$$

$$(\text{grad } f)_P = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]_P = [15, 3, -3] = 3[5, 1, -1]$$

Unit ~~vector~~ normal vector to level surface at pt P.

$$\hat{n} = \frac{(\text{grad } f)_P}{(|\text{grad } f|)_P} = \frac{3[5, 1, -1]}{3\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{3(5\vec{i} + \vec{j} - \vec{k})}{3\sqrt{27}}$$

$$\hat{n} = \frac{5\vec{i} + \vec{j} - \vec{k}}{3\sqrt{3}} \quad \underline{\text{Ans}}$$

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$$\textcircled{\frac{Q8}{78}} \quad x + y + z = 1, \quad P(4, 2, -5)$$

$$\text{Sol:-- } F = x + y + z$$

$$\frac{\partial F}{\partial x} = 1, \quad \frac{\partial F}{\partial y} = 1, \quad \frac{\partial F}{\partial z} = 1$$

$$(\text{grad } F)_P = \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]_P = [1, 1, 1]$$

$$|\text{grad } F| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Unit normal vector to the level surface at point P.

$$\hat{n} = \frac{(\text{grad } F)_P}{(|\text{grad } F|)_P} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} \quad \underline{\text{Ans}}$$

$\textcircled{\frac{Ex-3}{77}}$  Find unit normal vector at the point  $Q(2, 1, -1)$  to the level surface.

$$x^3 + y^3 + z^3 - 3xyz = 14.$$

$$\text{Sol: } f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$

$$\hat{n} = \frac{(\text{grad } f)_P}{|\text{grad } f|_P} \quad \textcircled{1}$$

$$\text{So, } \text{grad } f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3yz = 3(2)^2 - 3(1)(-1) = 15$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3xz = 3(1)^2 - 3(2)(-1) = 9$$

$$\frac{\partial F}{\partial z} = 3z^2 - 3xy = 3(-1)^2 - 3(2)(1) = -3$$

$$\hat{n} = \frac{15\vec{i} + 9\vec{j} - 3\vec{k}}{\sqrt{(15)^2 + (9)^2 + (-3)^2}} = \frac{3(5\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{9 \times 45}}$$

$$\hat{n} = \frac{5\vec{i} + 3\vec{j} - \vec{k}}{3\sqrt{5}} \quad \underline{\text{Ans}}$$

Q9  
78  $yz + zx + xy = 5, P(-1, 1, 1), \hat{n} = ?$

Sol:  $F(x, y, z) = yz + zx + xy$

$$\frac{\partial F}{\partial x} = z + y = 1 + 1 = 2$$

$$\frac{\partial F}{\partial y} = z + x = 1 - 1 = 0$$

$$\frac{\partial F}{\partial z} = y + x = 1 - 1 = 0$$

$$\hat{n} = \frac{2\vec{i} + 0\vec{j} + 0\vec{k}}{\sqrt{2^2 + 0^2 + 0^2}} = \frac{2\vec{i}}{2}$$

$$\hat{n} = \vec{i} \quad \underline{\text{Ans}}$$

Q10  
78  $x^2 + y^2 + z^2 = 6, P(1, 1, 1)$

Sol:  $F = x^2 + y^2 + z^2$

$$\frac{\partial F}{\partial x} = 2x = 2(1) = 2$$

$$\frac{\partial F}{\partial y} = 2y = 2(1) = 2$$

$$\frac{\partial F}{\partial z} = 2z = 2(1) = 2$$

$$\frac{(\text{grad } f)_p}{(|\text{grad } f|)_p} \quad (\text{grad } f)_p = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = [2, 2, 2]$$

$$(|\text{grad } f|)_p = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$$

Unit normal vector to the level surface at the given point.

$$\hat{n} = \frac{(\text{grad } f)_p}{(|\text{grad } f|)_p} = \frac{2\vec{i} + 2\vec{j} + 2\vec{k}}{2\sqrt{3}}$$

$$\hat{n} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} \quad \text{Ans}$$

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Theorem 75 Prove that  $\text{grad}(f+g) = \text{grad } f + \text{grad } g$ .

$$\text{Proof: } \text{grad}(f+g) = \vec{\nabla}(f+g)$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] (f+g)$$

$$= \left[ \frac{\partial}{\partial x} (f+g), \frac{\partial}{\partial y} (f+g), \frac{\partial}{\partial z} (f+g) \right]$$

$$= \left[ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \right]$$

$$= \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] + \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right]$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] f + \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] g$$

$$= \vec{\nabla} f + \vec{\nabla} g$$

$$\text{grad}(f+g) = \text{grad } f + \text{grad } g. \quad \text{proved.}$$

Theorem  
76

Prove that  $\text{grad}(fg) = f \text{grad} g + g \text{grad} f$

Proof:-  $\text{grad}(fg) = \vec{\nabla}(fg)$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] (fg) = \left[ \frac{\partial}{\partial x} (fg), \frac{\partial}{\partial y} (fg), \frac{\partial}{\partial z} (fg) \right]$$

$$= \left[ f \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x} g, f \frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} g, f \frac{\partial g}{\partial z} + \frac{\partial f}{\partial z} g \right]$$

$$= \left[ f \frac{\partial g}{\partial x}, f \frac{\partial g}{\partial y}, f \frac{\partial g}{\partial z} \right] + \left[ \frac{\partial f}{\partial x} g, \frac{\partial f}{\partial y} g, \frac{\partial f}{\partial z} g \right]$$

$$= f \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right] + g \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$= f \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] g + g \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] f$$

$$= f \vec{\nabla} g + g \vec{\nabla} f$$

$$\text{grad}(fg) = f \text{grad} g + g \text{grad} f \quad \text{proved}$$

Q12  
78

Prove that

$$(i) \text{grad}(fgh) = gh \text{grad} f + hf \text{grad} g + fg \text{grad} h$$

$$\text{Sol:- } \text{grad}(fgh) = \text{grad}[ \underline{f} (\underline{gh}) ]$$

$$= f \text{grad}(gh) + gh \text{grad} f$$

$$= f [ g \text{grad} h + h \text{grad} g ] + gh \text{grad} f$$

$$= fg \text{grad} h + fh \text{grad} g + gh \text{grad} f$$



rewriting in the reverse order we get  
 $\text{grad}(fgh) = gh \text{ grad } f + hf \text{ grad } g + fg \text{ grad } h.$

$$\text{ii) } \text{grad}\left(\frac{f}{g}\right) = \frac{g \text{ grad } f - f \text{ grad } g}{g^2}$$

$$\text{Sol: - } \text{grad}\left(\frac{f}{g}\right) = \text{grad } f (g)^{-1}$$

$$= f \text{ grad } g^{-1} + g^{-1} \text{ grad } f$$

$$= f \vec{\nabla} \cdot g^{-1} + \frac{1}{g} \text{ grad } f$$

$$= f \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] g^{-1} + \frac{1}{g} \text{ grad } f$$

$$= f \left[ \frac{\partial}{\partial x} g^{-1}, \frac{\partial}{\partial y} g^{-1}, \frac{\partial}{\partial z} g^{-1} \right] + \frac{1}{g} \text{ grad } f$$

$$= f \left[ -g^{-2} \frac{\partial g}{\partial x}, -g^{-2} \frac{\partial g}{\partial y}, -g^{-2} \frac{\partial g}{\partial z} \right] + \frac{1}{g} \text{ grad } f$$

$$= f \left( -\frac{1}{g^2} \right) \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] g + \frac{1}{g} \text{ grad } f$$

$$= -\frac{f}{g^2} \vec{\nabla} g + \frac{1}{g} \text{ grad } f$$

$$= -\frac{f}{g^2} \text{ grad } g + \frac{1}{g} \text{ grad } f$$

$$= \frac{-f \text{ grad } g + g \text{ grad } f}{g^2}$$

$$\text{grad}\left(\frac{f}{g}\right) = \frac{g \text{ grad } f - f \text{ grad } g}{g^2}$$

proved

Ex-4  
77

If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  then evaluate  $\text{grad } r^n$ .

Sol:  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$$

$$r^n = \left[ (x^2 + y^2 + z^2)^{1/2} \right]^n = (x^2 + y^2 + z^2)^{n/2}$$

$$\text{grad } r^n = \text{grad} (x^2 + y^2 + z^2)^{n/2} = \vec{\nabla} (x^2 + y^2 + z^2)^{n/2}$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] (x^2 + y^2 + z^2)^{n/2}$$

$$= \left[ \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{n/2}, \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{n/2}, \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{n/2} \right]$$

$$= \left[ \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n}{2}-1} (2x), \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n}{2}-1} (2y), \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n}{2}-1} (2z) \right]$$

$$= n (x^2 + y^2 + z^2)^{\frac{n}{2}-1} [x, y, z]$$

$$= n (x^2 + y^2 + z^2)^{\frac{n-2}{2}} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= n \left[ (x^2 + y^2 + z^2)^{1/2} \right]^{n-2} \vec{r}$$

$$= n r^{n-2} \vec{r} \quad \because (x^2 + y^2 + z^2)^{1/2} = r$$

$$\boxed{\text{grad } r^n = n r^{n-2} \vec{r}}$$

proved

(ii)  $\frac{79}{79}$  prove  $\text{div}(c\vec{F}) = c \text{div} F \Rightarrow c$  is constant.

Sol:- let  $\vec{F} = [f_1, f_2, f_3]$

$$\text{div}(c\vec{F}) = \vec{\nabla} \cdot c[f_1, f_2, f_3]$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [cf_1, cf_2, cf_3]$$

$$= \frac{\partial}{\partial x} cf_1 + \frac{\partial}{\partial y} cf_2 + \frac{\partial}{\partial z} cf_3$$

$$= c \left[ \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z} \right]$$

$$= c \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [f_1, f_2, f_3]$$

$$\text{div}(c\vec{F}) = c \text{div} \vec{F}, \text{ proved.}$$

(iii)  $\frac{79}{79}$  prove  $\text{div}(\phi\vec{F}) = \phi \text{div} F + \text{grad} \phi \cdot \vec{F}$ .

let  $\vec{F} = [f_1, f_2, f_3]$

$$\text{div}(\phi\vec{F}) = \vec{\nabla} \cdot \phi\vec{F} = \vec{\nabla} \cdot \phi[f_1, f_2, f_3]$$

$$= \vec{\nabla} \cdot [\phi f_1, \phi f_2, \phi f_3] = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [\phi f_1, \phi f_2, \phi f_3]$$

$$= \frac{\partial}{\partial x} (\phi f_1) + \frac{\partial}{\partial y} (\phi f_2) + \frac{\partial}{\partial z} (\phi f_3)$$

$$= \phi \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial x} f_1 + \phi \frac{\partial f_2}{\partial y} + \frac{\partial \phi}{\partial y} f_2 + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial z} f_3$$

$$= \phi \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial x} f_1 + \phi \frac{\partial f_2}{\partial y} + \frac{\partial \phi}{\partial y} f_2 + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial z} f_3$$

$$\begin{aligned} \operatorname{div}(\phi \vec{F}) &= \phi \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) + \left[ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right] \cdot [f_1, f_2, f_3] \\ &= \phi \vec{\nabla} \cdot \vec{F} + \vec{\nabla} \phi \cdot \vec{F} \end{aligned}$$

$$\operatorname{div}(\phi \vec{F}) = \phi \operatorname{div} \vec{F} + \operatorname{grad} \phi \cdot \vec{F} \quad \text{proved.}$$

## CURL OF A VECTOR

Let  $\vec{F} = [f_1, f_2, f_3]$  be a vector, Then

$\vec{\nabla} \times \vec{F}$  is called curl of  $\vec{F}$  and denoted by

$\operatorname{Curl} \vec{F}$ .

$$\operatorname{Curl} \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

(EX 83)

Calculate  $\operatorname{Curl} \vec{F}$  at  $(1, 1, 1)$  if

$$F(x, y, z) = x^2 y \vec{i} + (4xz + y^2) \vec{j} + (5z^2 + xy^2) \vec{k}$$

$$\text{Sol: } \operatorname{Curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 4xz + y^2 & 5z^2 + xy^2 \end{vmatrix}$$

By R<sub>1</sub>

$$\operatorname{Curl} \vec{F} = \vec{i} \left[ \frac{\partial}{\partial y} (5z^2 + xy^2) - \frac{\partial}{\partial z} (4xz + y^2) \right]$$

$$\begin{aligned}
 & + \vec{j} \left[ \frac{\partial}{\partial z} (x^2 - y) - \frac{\partial}{\partial x} (5z^2 + xy^2) \right] \\
 & + \vec{k} \left[ \frac{\partial}{\partial x} (4x(z+y^2)) - \frac{\partial}{\partial y} (x^2y) \right] \\
 = & \vec{i} [(0+2xy) - (4x+0)] + \vec{j} [0 - (0+y^2)] \\
 & + \vec{k} [(4z+0) - x^2] \\
 & \text{at } p(1, 1, 1).
 \end{aligned}$$

$$(\text{Curl } \vec{F})_p = (2 \cdot 1 - 4 \cdot 1) \vec{i} + \vec{j} (-(1)^2) + \vec{k} (4 \cdot 1 - 1^2)$$

$$(\text{Curl } \vec{F})_p = -2\vec{i} - \vec{j} + 3\vec{k} \quad \underline{\text{Ans}}$$

### EXERCISE #5.4

Calculate  $\text{Curl } \vec{F}$   $\nabla \times \vec{F}$

$$\text{(Q1)} \quad \vec{F} = (y^2 + z^2) \vec{i} + (z^2 + x^2) \vec{j} + (x^2 + y^2) \vec{k}$$

$$\text{Sol: } \text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & z^2 + x^2 & x^2 + y^2 \end{vmatrix}$$

$$\begin{aligned}
 = & \vec{i} \left[ \frac{\partial}{\partial y} (x^2 + y^2) - \frac{\partial}{\partial z} (z^2 + x^2) \right] + \vec{j} \left[ \frac{\partial}{\partial z} (y^2 + z^2) - \frac{\partial}{\partial x} (x^2 + y^2) \right] \\
 & + \vec{k} \left[ \frac{\partial}{\partial x} (z^2 + x^2) - \frac{\partial}{\partial y} (y^2 + z^2) \right] \\
 = & \vec{i} [2y - 2z] + \vec{j} [2z - 2x] + \vec{k} [2x - 2y]
 \end{aligned}$$

$$\text{Curl } \vec{F} = 2(y-z)\vec{i} + 2(z-x)\vec{j} + 2(x-y)\vec{k} \quad \underline{\text{Ans}}$$

$$\textcircled{Q 2} \vec{F} = (x^2 + y^2 + z^2)^{-\frac{1}{2}} (y z \vec{i} + z x \vec{j} + x y \vec{k})$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2 + z^2)^{-\frac{1}{2}} y z & (x^2 + y^2 + z^2)^{-\frac{1}{2}} z x & (x^2 + y^2 + z^2)^{-\frac{1}{2}} x y \end{vmatrix}$$

$$= \vec{i} \left[ \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-\frac{1}{2}} x y - \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-\frac{1}{2}} y z \right]$$

$$+ \vec{j} \left[ \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-\frac{1}{2}} y z + \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-\frac{1}{2}} x y \right]$$

$$+ \vec{k} \left[ \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-\frac{1}{2}} z x - \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-\frac{1}{2}} x y \right]$$

$$= \vec{i} \left[ x \left\{ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 1 + y \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2y) \right\} \right. \\ \left. - y \left\{ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 1 + z \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2z) \right\} \right]$$

$$+ \vec{j} \left[ x \left\{ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 1 + z \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2z) \right\} \right. \\ \left. - y \left\{ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 1 + x \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2x) \right\} \right]$$

$$+ \vec{k} \left[ z \left\{ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 1 + x \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2x) \right\} \right. \\ \left. - z \left\{ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 1 + y \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} (2y) \right\} \right]$$

$$= \vec{i} (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[ x \{ x^2 + y^2 + z^2 - y^2 \} - y \{ x^2 + y^2 + z^2 - z^2 \} \right]$$

$$+ \vec{j} (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[ y \{ x^2 + y^2 + z^2 - z^2 \} - y \{ x^2 + y^2 + z^2 - x^2 \} \right]$$

$$+ \vec{k} (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[ z \{ x^2 + y^2 + z^2 \} - x^2 \{ -z \{ x^2 + y^2 + z^2 - y^2 \} \} \right]$$

$$= (x^2 + y^2 + z^2)^{-3/2} \left[ x(x^2 + z^2 - x^2 - y^2) \vec{i} + y(x^2 + y^2 - y^2 - z^2) \vec{j} + z(y^2 + z^2 - x^2 - z^2) \vec{k} \right]$$

$$= (x^2 + y^2 + z^2)^{-3/2} \left[ x(z^2 - y^2) \vec{i} + y(x^2 - z^2) \vec{j} + z(y^2 - x^2) \vec{k} \right]$$

Ans

(Q 3 / 84)  $\vec{F} = yz \cos x \vec{i} + zx \cos y \vec{j} + xy \cos z \vec{k}$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz \cos x & zx \cos y & xy \cos z \end{vmatrix}$$

$$= \vec{i} \left[ \frac{\partial}{\partial y} (xy \cos z) - \frac{\partial}{\partial z} (zx \cos y) \right] + \vec{j} \left[ \frac{\partial}{\partial z} (yz \cos x) - \frac{\partial}{\partial x} (xy \cos z) \right] + \vec{k} \left[ \frac{\partial}{\partial x} (zx \cos y) - \frac{\partial}{\partial y} (yz \cos x) \right]$$

$$= \vec{i} [x \cos z - x \cos y] + \vec{j} [y \cos x - y \cos z] + \vec{k} [z \cos y - z \cos x]$$

$$\text{Curl } \vec{F} = x(z \cos z - \cos y) \vec{i} + y(\cos x - \cos z) \vec{j} + z(\cos y - \cos x) \vec{k}$$

Ans

$$\textcircled{Q4} \quad \vec{F} = x(y^2+z^2)\vec{i} + y(z^2+x^2)\vec{j} + z(x^2+y^2)\vec{k}$$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(y^2+z^2) & y(z^2+x^2) & z(x^2+y^2) \end{vmatrix}$$

$$= \vec{i} \left[ \frac{\partial}{\partial y} z(x^2+y^2) - \frac{\partial}{\partial z} y(z^2+x^2) \right] + \vec{j} \left[ \frac{\partial}{\partial x} x(y^2+z^2) - \frac{\partial}{\partial x} z(x^2+y^2) \right] + \vec{k} \left[ \frac{\partial}{\partial x} y(z^2+x^2) - \frac{\partial}{\partial y} x(y^2+z^2) \right]$$

$$\text{Curl } \vec{F} = \left[ z(2y) - y(2z) \right] \vec{i} + \left[ x(y^2+z^2) - z(2x+y^2) \right] \vec{j} + \left[ y(z^2+2x) - x(2y+z^2) \right] \vec{k}$$

$$= \left[ x^2z + 2yz - 2yz - x^2y \right] \vec{i} + \left[ xy^2 + xz^2 - 2xz - 2yz \right] \vec{j} + \left[ yz^2 + 2xy - 2xy - xz^2 \right] \vec{k}$$

$$\text{Curl } \vec{F} = x^2(z-y)\vec{i} + y^2(x-z)\vec{j} + z^2(y-x)\vec{k}$$

Ans



$$\textcircled{\frac{Q5}{84}} \vec{F} = yz \log x \vec{i} + zx \log y \vec{j} + xy \log z \vec{k}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz \log x & zx \log y & xy \log z \end{vmatrix}$$

$$= \vec{i} \left[ \frac{\partial}{\partial y} (xy \log z) - \frac{\partial}{\partial z} (zx \log y) \right]$$

$$+ \vec{j} \left[ \frac{\partial}{\partial z} (yz \log x) - \frac{\partial}{\partial x} (xy \log z) \right]$$

$$+ \vec{k} \left[ \frac{\partial}{\partial x} (zx \log y) - \frac{\partial}{\partial y} (yz \log x) \right]$$

$$= [x \log z - x \log y] \vec{i} + [y \log x - y \log z] \vec{j}$$

$$+ [z \log y - z \log x] \vec{k}$$

$$= x(\log z - \log y) \vec{i} + y(\log x - \log z) \vec{j}$$

$$+ z(\log y - \log x) \vec{k}$$

$$\text{Curl } \vec{F} = x \log \frac{z}{y} \vec{i} + y \log \frac{x}{z} \vec{j} + z \log \frac{y}{x} \vec{k}$$

Ans

v.v. imp  
VII  
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Ch-05

(42)

Prove  $\vec{\nabla} \times (b\vec{F} + c\vec{g}) = b\vec{\nabla} \times \vec{F} + c\vec{\nabla} \times \vec{g}$

Proof:  $\vec{F} = [f_1, f_2, f_3], \vec{g} = [g_1, g_2, g_3]$

$$\vec{\nabla} \times (b\vec{F} + c\vec{g}) = \vec{\nabla} \times [b[f_1, f_2, f_3] + c[g_1, g_2, g_3]]$$

$$= \vec{\nabla} \times [ [bf_1, bf_2, bf_3] + [cg_1, cg_2, cg_3] ]$$

$$= \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \times [bf_1 + cg_1, bf_2 + cg_2, bf_3 + cg_3]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bf_1 + cg_1 & bf_2 + cg_2 & bf_3 + cg_3 \end{vmatrix} \quad \text{By R}_1$$

$$= \vec{i} \left[ \frac{\partial}{\partial y} (bf_3 + cg_3) - \frac{\partial}{\partial z} (bf_2 + cg_2) \right] + \vec{j} \left[ \frac{\partial}{\partial z} (bf_1 + cg_1) - \frac{\partial}{\partial x} (bf_3 + cg_3) \right] + \vec{k} \left[ \frac{\partial}{\partial x} (bf_2 + cg_2) - \frac{\partial}{\partial y} (bf_1 + cg_1) \right]$$

$$= \vec{i} \left[ b \frac{\partial f_3}{\partial y} + c \frac{\partial g_3}{\partial y} - b \frac{\partial f_2}{\partial z} - c \frac{\partial g_2}{\partial z} \right]$$

$$+ \vec{j} \left[ b \frac{\partial f_1}{\partial z} + c \frac{\partial g_1}{\partial z} - b \frac{\partial f_3}{\partial x} - c \frac{\partial g_3}{\partial x} \right]$$

$$+ \vec{k} \left[ b \frac{\partial f_2}{\partial x} + c \frac{\partial g_2}{\partial x} - b \frac{\partial f_1}{\partial y} - c \frac{\partial g_1}{\partial y} \right]$$

$$= \vec{i} \left[ b \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \right] + \vec{j} \left[ c \left( \frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z} \right) \right]$$

$$+ \vec{j} \left[ b \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \right] + \vec{j} \left[ c \left( \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x} \right) \right]$$

$$+ \vec{k} \left[ b \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \right] + \vec{k} \left[ c \left( \frac{\partial g_1}{\partial z} - \frac{\partial g_1}{\partial x} \right) \right]$$

$$= b \left[ \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \vec{i} + \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \vec{j} + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \vec{k} \right]$$

$$+ c \left[ \left( \frac{\partial g_1}{\partial y} - \frac{\partial g_2}{\partial z} \right) \vec{i} + \left( \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x} \right) \vec{j} + \left( \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial x} \right) \vec{k} \right]$$

$$\vec{\nabla} \times (b \vec{F} + c \vec{g}) = b \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} + c \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$\vec{\nabla} \times (b \vec{F} + c \vec{g}) = b \vec{\nabla} \times \vec{F} + c \vec{\nabla} \times \vec{g}. \text{ proved}$$

viii) prove  $\vec{\nabla} \times (\phi \vec{F}) = \phi \vec{\nabla} \times \vec{F} + (\vec{\nabla} \phi) \times \vec{F}$ .

Proof:-  $\vec{F} = [f_1, f_2, f_3]$

$$\vec{\nabla} \times (\phi \vec{F}) = \vec{\nabla} \times [\phi [f_1, f_2, f_3]].$$

$$= \vec{\nabla} \times [\phi f_1, \phi f_2, \phi f_3]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi f_1 & \phi f_2 & \phi f_3 \end{vmatrix}$$

$$= \vec{i} \left[ \frac{\partial}{\partial y} (\phi f_3) - \frac{\partial}{\partial z} (\phi f_2) \right] + \vec{j} \left[ \frac{\partial}{\partial z} (\phi f_1) - \frac{\partial}{\partial x} (\phi f_3) \right] + \vec{k} \left[ \frac{\partial}{\partial x} (\phi f_2) - \frac{\partial}{\partial y} (\phi f_1) \right]$$

$$= \vec{i} \left[ \phi \frac{\partial f_3}{\partial y} + \frac{\partial \phi}{\partial y} f_3 - \phi \frac{\partial f_2}{\partial z} - \frac{\partial \phi}{\partial z} f_2 \right] + \vec{j} \left[ \phi \frac{\partial f_1}{\partial z} + \frac{\partial \phi}{\partial z} f_1 - \phi \frac{\partial f_3}{\partial x} - \frac{\partial \phi}{\partial x} f_3 \right] + \vec{k} \left[ \phi \frac{\partial f_2}{\partial x} + \frac{\partial \phi}{\partial x} f_2 - \phi \frac{\partial f_1}{\partial y} - \frac{\partial \phi}{\partial y} f_1 \right]$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \vec{i} \left[ \frac{\partial}{\partial y} f_3 - \frac{\partial f_2}{\partial z} \right] + \vec{j} \left[ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right] + \vec{k} \left[ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right]$$

$$\phi \vec{\nabla} \times \vec{F} = \vec{i} \left[ \phi \frac{\partial f_3}{\partial y} - \phi \frac{\partial f_2}{\partial z} \right] + \vec{j} \left[ \phi \frac{\partial f_1}{\partial z} - \phi \frac{\partial f_3}{\partial x} \right] + \vec{k} \left[ \phi \frac{\partial f_2}{\partial x} - \phi \frac{\partial f_1}{\partial y} \right]$$

$$(\vec{\nabla} \phi) \times \vec{F} = \left[ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right] \times [f_1, f_2, f_3]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$(\vec{\nabla}\phi) \times \vec{F} = \vec{i} \left[ f_3 \frac{\partial \phi}{\partial y} - f_2 \frac{\partial \phi}{\partial z} \right] + \vec{j} \left[ f_1 \frac{\partial \phi}{\partial z} - f_3 \frac{\partial \phi}{\partial x} \right] \\ + \vec{k} \left[ f_2 \frac{\partial \phi}{\partial x} - f_1 \frac{\partial \phi}{\partial y} \right] \quad (3)$$

$$(2) + (3)$$

$$\phi \vec{\nabla} \times \vec{F} + (\vec{\nabla}\phi) \times \vec{F} = \vec{i} \left[ \phi \frac{\partial f_3}{\partial y} + \frac{\partial \phi}{\partial y} f_3 - \phi \frac{\partial f_2}{\partial z} - \frac{\partial \phi}{\partial z} f_2 \right] \\ + \vec{j} \left[ \phi \frac{\partial f_1}{\partial z} + \frac{\partial \phi}{\partial z} f_1 - \phi \frac{\partial f_3}{\partial x} - \frac{\partial \phi}{\partial x} f_3 \right] \\ + \vec{k} \left[ \phi \frac{\partial f_2}{\partial x} + \frac{\partial \phi}{\partial x} f_2 - \phi \frac{\partial f_1}{\partial y} - \frac{\partial \phi}{\partial y} f_1 \right] \quad (4)$$

$$(1) = (4)$$

$$\therefore \vec{\nabla} \times (\phi \vec{F}) = \phi \vec{\nabla} \times \vec{F} + (\vec{\nabla}\phi) \times \vec{F} \quad \text{proved}$$